SIZE DIRECTION GAMES OVER THE REAL LINE. III

BY

GADI MORAN AND SAHARON SHELAH

ABSTRACT

We prove, using the continuum hypothesis, that D (the direction player) has a winning strategy in $\tilde{\Gamma}^S_Q(X)$ for some uncountable X, and that there is an uncountable X which intersects each perfect nowhere-dense set of reals in a countable set such that D does not win in ${}_a\Gamma^S_Q(X)$ for every a. We also give another proof to the fact that $\Gamma^S(X)$ is a win for D if X is countable.

0 Introduction

This paper continues Ehrenfeucht and Moran [1] and Moran [3]. We use the notations of [3]. In addition to the results mentioned in the abstract, we prove a technical lemma. This lemma is a theorem of ZF (even AC is not needed).

In [3, Th. 5.8] it is proved that if $\Gamma^S_Q(X)$ is a win for D then X is at most denumerable. Here in Theorem 2.1, we prove assuming CH that there is an uncountable X such that $\tilde{\Gamma}^S_Q(X)$ is a win for D. By [3, Th. 5.5], this is a theorem of ZF for ${}_a\Gamma^S_Q(X)$. Yet, for $\tilde{\Gamma}^S(X)$ and even ${}_a\Gamma^S(X)$, this is still an open question.

By [1, Th. 2], if X is countable then $\Gamma^S(X)$ (hence also $\tilde{\Gamma}^S_Q(X)$) is a win for D. We give here another proof to this effect, which has the flavor of the priority method (Theorem 3.1). In Theorem 2.2 we prove, assuming CH, that this does not generalize to uncountable X intersecting each perfect nowhere-dense set by a countable set, for the game $\tilde{\Gamma}^S_Q(X)$.

By Solovay [4], Theorem 2.1 cannot be proved using ZF only. It is an open question whether we can prove it in ZFC or even in ZFC+MA (on Martin's axiom MA, see Martin and Solovay [2].)

Theorem 2.2 generalizes to the case $2^{\aleph_0} > \aleph_1$, if the union of $< 2^{\aleph_0}$ sets of first category is a set of the first category (this is a conclusion of MA; see [2]) as

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follows: There is a set X whose intersection with every perfect nowhere-dense set is of power $< 2^{\aleph_0}$, but D has no winning strategy in ${}_a\Gamma_Q^S(X)$. It is an open question whether Theorem 2.2 as stated can be proved in ZFC or even ZFC+ MA.

NOTATION. We use the notation of [3]. We remind the reader that a bis is a function J defined on 2^* such that $J(\xi)$ is a nonempty closed interval for $\xi \in 2^*$, $J(\xi \cdot \langle 0 \rangle)$, $J(\xi \cdot \langle 1 \rangle) \subseteq J(\xi)$ and $J(\xi \cdot \langle 0 \rangle) < J(\xi \cdot \langle 1 \rangle)$. For $\xi \in 2^*$, J_ξ is the bis defined by $J_\xi(\eta) = J(\xi \cdot \eta)$, $\eta \in 2^*$. KJ denotes the perfect set ${}_n\bigcap_{<\omega} \bigcup_{l\xi = n} J(\xi)$. $\phi: 2^* \to 2^*$ is an embedding if for every $\xi, \xi' \in 2^*$, $\xi \prec \xi'$ iff $\phi(\xi) \prec \phi(\xi')$. A bis J' *refines* a bis J if there is an embedding ϕ: $2^* \to 2^*$ such that $J'(\xi) \subseteq J(\phi(\xi))$. It is clear that if J' refines J then $KJ' \subseteq KJ$. A *con* is a nonincreasing sequence of positive numbers that converges to zero. A sequence $\langle s_n \colon n < \rangle$ of real numbers *obeys* a con $\langle a_n \colon n < \omega \rangle$ if for $m, m' \geqq n$, $|s_m - s_{m'}| < a_n$.

1. How D escapes a countable number of threats

THEOREM 1.1. *Let $\boldsymbol{a} = \langle a_n \colon n < \omega \rangle$ be a con, and let J_n be a bis, z_n a real number, $n \in \omega$. Then for every $n \in \omega$, there is a bis J_n' so that J_n' refines J_n, and if*

$$X = \{z_n \colon n < \omega\} \cup \bigcup_{n < \omega} KJ_n'$$

then ${}_a\Gamma_Q^S(X)$ is a win for D.

PROOF. We shall first define by simultaneous induction J_n' for $n \in \omega$, and then describe a winning strategy for D in ${}_a\Gamma_Q^S(X)$.

I. Let $Q = \{q_n \colon n < \omega\}$ be any enumeration of Q. Following the proof of [3, Th. 5.5], we define by induction on n $\phi_i(\xi)$, $J_i'(\xi)$ and $k_n \in \omega$, for ξ, i satisfying $\max\{l\xi, i\} = n$ so that the following requirements will hold.

Let $F_n = \bigcup_{0 \leqq i \leqq n} \bigcup_{\xi \in 2^n} J_i'(\xi)$.

0) $a_{k_n} < mg$ for every component g of $R - F_n$.

1) $q_n \notin T^{k_n} F_n$, (see [3, Definition 3.0]).

2) $J_i'(\xi) = J_i(\phi_i(\xi))$.

Let $\xi_0 \in 2^*$ satisfy: $q_0 \notin J_0(\xi_0)$. Define: $k_0 = 0$, $\phi_0(\varnothing) = \xi_0$, $J_0'(\varnothing) = J_0(\xi_0)$.

Assume that $n > 0$ and that ϕ_i, $J_i(\xi)$ are defined for $\xi \in \bigcup_{m<n} 2^m$, $i < n$. Let $\phi_n(\xi) = \xi$, $J_n'(\xi) = J_n(\xi)$ for $\xi \in \bigcup_{m<n} 2^m$, and let $\xi_0, \cdots, \xi_{2^{n-1}-1}$ be the enumeration of 2^{n-1} in the lexicographical ordering. We have for $0 \leqq j < j' < 2^{n-1}$:

$$J_i(\xi_j) < J_i'(\xi_{j'}).$$

Recall that $H(X)$ denotes the set of all $s \in R$ such that for every $\varepsilon > 0$ $(s - \varepsilon, s) \cap X \neq \varnothing$ and $(s, s + \varepsilon) \cap X \neq \varnothing$ [3, Definition 4.1]; see also Lemma 4.2 (ii)). Let $S_n = \{s_{ij} \colon 0 \leqq i \leqq n, 0 \leqq j < 2^n\}$ satisfy:

3) $s_{i,j} \in KJ_i$, $0 \leqq j < 2^n$

4) $s_{i,j} < s_{i,j'}$, $0 \leqq i \leqq n$, $0 \leqq j < j' < 2^n$

5) $s_{i,2j}, s_{i,2j+1} \in J_i'(\xi_j)$, $0 \leqq i \leqq n$, $0 \leqq j < 2^{n-1}$

6) S_n is linearly independent over Q.

From (6) it follows [3, Proposition 3.4] that $T^\omega S_n \cap Q = \emptyset$. Define $d_n > 0$ by:

7) $3d_n = \min\{|s' - s''| : s', s'' \in S_n, s' \neq s''\}$.

Let $k_n \in \omega$ be the least k such that $a_k < d_n$. Define $\delta > 0$ by:

8) $2\delta = \min\{d_n, \min\{|q_n - s| : s \in T^{k_n}S_n\}\}$.

Thus, $q_n \notin \bigcup_{s \in T^{k_n}S_n}[s - \delta, s + \delta]$. Hence [3, Lemma 3.5]

9) $q_n \notin T^{k_n}(\bigcup_{s \in S_n}[s - \delta, s + \delta])$.

Let $0 \leqq j < 2^{n-1}$, $0 \leqq i \leqq n$ be given. Then $s_{i,2j} s_{i,2j+1} \in H(KJ_i) \cap J_i(\phi_i(\xi_j))$; hence, it is possible to find ξ_j^0, ξ_j^1 so that $\xi_j \prec \xi_j^\varepsilon$ for $\varepsilon \in 2$, $J_i(\xi_j^0) < J_i(\xi_j^1)$, $s_{i,2j+\varepsilon} \in J_i(\xi_j^\varepsilon)$, $\varepsilon \in 2$, and also $m\, J_i(\xi_j^\varepsilon) \leqq \delta$. It follows that

10) $J_i(\xi_j^\varepsilon) \subset [s_{i,2j+\varepsilon} - \delta, s_{i,2j+\varepsilon} + \delta]$.

We define at last:

$$\phi_i(\xi_j \cdot \langle \varepsilon \rangle) = \xi_j^\varepsilon,\ J_i'(\xi_j \langle \varepsilon \rangle) = J_i(\xi_j^\varepsilon), \qquad \varepsilon \in 2;$$

thus, (2) is already satisfied. By (9), (10) we see that (1) is also satisfied. From (7), (8) it follows that (0) is satisfied. This completes the construction of J_n' for all $n < \omega$.

II. We shall now describe a winning strategy for D in ${}_a\Gamma_Q^S(X)$ (compare [3, Lemma 5.3, Lemma 5.1]).

Step 0. (i)$_0$ Assume that S started with $s_0 = q_{n_0}$. By (1) and [3, Th. 3.2], $G(F_{n_0}; q_{n_0}; k_{n_0})$ is a win for D. D follows his winning strategy k_{n_0} moves. This ensures that $s_{k_{n_0}} \notin F_{n_0}$; hence, $d(s_{k_{n_0}}, F_{n_0}) > 0$.

(ii)$_0$ D picks $r_0 > k_{n_0}$ so that $a_{r_0} \leqq d(s_{k_{n_0}}, F_{n_0})$ and makes sure by recoiling from F_{n_0}, that if a is not violated, then $g_0 = (s_{r_0} - a_{r_0}, s_{r_0} + a_{r_0}) \cap (s_{k_{n_0}} - a_{k_{n_0}}, s_{k_{n_0}} + a_{k_{n_0}})$ has a positive distance from F_{n_0}. This is possible by (0). Note that if a is not to be violated, s_m should belong to g_0 for $m \geqq r_0$, and hence the outcome cannot belong to F_{n_0}.

(iii)$_0$ D makes (if necessary) the distance δ_0 from $s_{r_0} + 1$ to z_0 positive, and picks $t_0 > r_0$ so that $a_{t_0} < \delta_0$. Then he recoils from z_0 until s_{t_0} is constructed. Thus it ensures that if a will not be violated, the outcome will be different from z_0.

The j'th step is essentially the same as the first. The play, where a is not yet violated, reached $s_{t_{j-1}} = q_{n_j} \in Q$. As in step 0:

$(\mathrm{i})_j$ D makes sure that $s_{k_j} \notin F_{n_j}$, where $k_{n_j} \leqq k'_j$ (use (1)).

$(\mathrm{ii})_j$ D makes sure that s_{r_j} belongs to an open interval g_j where the rest of the play, if $\boldsymbol{a}$ is not to be violated, should fall, and such that g_j has a positive distance from F_{n_j} (use (0)). The outcome cannot lie in F_{n_j} unless $\boldsymbol{a}$ is violated.

$(\mathrm{iii})_j$ D makes sure that s_{t_j} is in an interval of positive distance from z_j, where all the next elements of the play should fall, unless $\boldsymbol{a}$ is violated.

We show now that this is a winning strategy. Indeed, if $\boldsymbol{s}$ is a play where D follows this strategy, then by $(\mathrm{iii})_j$, the outcome s is not equal to z_j for any $j < \omega$. Also, by $(\mathrm{ii})_j$, the outcome s does not belong to infinitely many F_n's. It follows that if $i \in \omega$, then $s \notin KJ'_i$. This is clear, since:

$$KJ_i \subset \bigcap_{i \leqq n} F_n$$

2.

D rides CH and escapes an uncountable number of points

THEOREM 2.1 (CH) *In every perfect set there is an uncountable set X s.t. $\tilde{\Gamma}^S_Q(X)$ is a win for D.*

PROOF. Let $\langle \boldsymbol{a}^\alpha : \alpha < \omega_1 \rangle$ be a list of all the cons, i.e., nonincreasing sequences of positive numbers that converge to zero. Let P be a perfect set, and let J^* be any bis such that $KJ^* \subseteq P$, (see [3, Lemma 4.2]). We now define by induction on α:

1) a countable family $\mathscr{J}^\alpha$ of refinements of J^*. We put $A_\alpha = \bigcup_{J \in \mathscr{J}^\alpha} KJ$.

2) for each ordinal α, a real x_α so that

A) if $\beta < \alpha$, $J \in \mathscr{J}^\beta$, $\xi \in 2^*$, *then* there is a $J' \in \mathscr{J}$ such that J' refines J_ξ.

B) $x_\alpha \in A_\alpha - \{x_\beta : \beta < \alpha\}$.

C) ${}_{\boldsymbol{a}^\alpha}\Gamma^S_Q(A_{\alpha+1} \cup \{x_\beta : \beta \leqq \alpha\})$ is a win for D.

D) $A_\alpha \supsetneqq A_\beta$ for $\beta < \alpha$.

It is clear that if the induction can be carried out, we are done. We put $X = \{x_\alpha : \alpha < \omega_1\}$. Then X is uncountable, and D has a winning strategy in $\tilde{\Gamma}^Q_S(X)$. Denote by τ_α the winning strategy ensured by (C) in the $\alpha + 1$'th step of the induction. If S chooses $\boldsymbol{a}^\alpha$ as a con, D uses τ_α and wins, put $\mathscr{J}^0 = \{J^*\}$, $A_0 = KJ^*$.

Assume that $\mathscr{J}^\beta$, A_β, x_β are defined for $\beta < \alpha$ so that (A)–(D) hold.

Case I. α is a successor.

Assume that $\alpha = \gamma + 1$. The set $\{J'_\xi : J' \in \mathscr{J}^\gamma, \xi \in 2^*\}$ is denumerable, and so is $\{x_\beta : \beta \leqq \gamma\}$; use Theorem 1.1, to obtain a countable family $\mathscr{J}^\alpha$ so that for each $\xi \in 2^*$, $J' \in \mathscr{J}^\gamma$, there is a $J \in \mathscr{J}^\alpha$ so that J refines J'_ξ, and ${}_{\boldsymbol{a}^\alpha}\Gamma_Q(A^S_\alpha \bigcup \{x_\beta : \beta < \alpha\})$

is a win for D (where A_α is defined in (1)). Pick $x_\alpha \in A_\alpha$ so that $x_\alpha \neq x_\beta$ for $\beta < \alpha$ (A_α has the power of the continuum). It is clear that by the induction hypothesis, (A)–(D) are carried over.

Case II. α is a limit ordinal.

Assume that $\beta < \alpha$, $\xi \in 2^*$, $J' \in \mathscr{J}^\beta$.

Let $\langle \beta_n : n < \omega \rangle$ be an increasing sequence of ordinals whose limit is α, $\beta_0 = \beta$. We shall define a refinement J of J', and a bis J^ζ for $\zeta \in 2^*$ by induction on $l\zeta$ so that:

a) if $l\zeta = n$ then $J^\zeta \in \mathscr{J}^{\beta_n}$

b) if $\zeta \prec \zeta'$ then $J^{\zeta'}$ refines J^ζ

c) $J(\zeta) = J^\zeta(\zeta)$.

Put:

$$J(\varnothing) = J'_\xi(\varnothing), \; J^\alpha = J'_\xi$$

Assume that $J(\zeta)$, J^ζ are defined for $\zeta \in 2^n$ so that (a)–(c) hold. Use the induction hypothesis (A) to pick $J^{\zeta\langle\varepsilon\rangle} \in \mathscr{J}^{\beta_{n+1}}$ so that $J^{\zeta\langle\varepsilon\rangle}$ refines $J^\zeta_{\langle\varepsilon\rangle}$, $\varepsilon \in 2$, and define $J(\zeta \cdot \langle\varepsilon\rangle)$ by (c).

It is clear that for every $\alpha \in 2^\omega$, $\bigcap_{n \leqq \omega} J(\bar{\alpha}(n)) = \bigcap_{n < \omega} J^{\bar{\alpha}(n)}(\bar{\alpha}(n)) \in KJ^{\bar{\alpha}(m)}$ for all $m < \omega$ by (b); hence, $KJ \subseteq \bigcap_{\beta < \alpha} A_\beta$. Thus, also $A_\alpha \subseteq A_\beta$ for $\beta < \alpha$. So (D) holds.

Now choose $x_\alpha \in A_\alpha - \{x_\beta : \beta < \alpha\}$.

It is clear that (A) holds. Both (B) and (C) hold vacuously, and (D) is easy. □

THEOREM 2.2. (CH) *There is a set X of points such that*

(i) *X is uncountable, but for each perfect nowhere-dense set P $|P \cap X| \leqq \aleph_0$,*

(ii) *for no $\boldsymbol{a}$, D has a winning strategy in the game ${}_{\boldsymbol{a}}\Gamma^S_Q(X)$.*

PROOF. Let $\langle P_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of all perfect nowhere-dense sets, and let $\{\langle \boldsymbol{a}^\alpha, \tau^\alpha \rangle : \alpha < \omega_1\}$ be an enumerativn of all pairs $(\boldsymbol{a}, \tau)$ where $\boldsymbol{a}$ is a con, τ a strategy of D in the game ${}_{\boldsymbol{a}}\Gamma^S_Q$.

We define by induction on $\alpha < \omega_1$ $x^\alpha \in R$ such that

1) $x^\alpha \notin \bigcup_{\beta \leqq \alpha} P_\beta$

2) there is a play of ${}_{\boldsymbol{a}}\alpha\Gamma^S_Q$ in which D uses the strategy τ^α and the outcome is x^α.

In order to carry the induction we need to show only that in ${}_{\boldsymbol{a}^\alpha}\Gamma^S_Q(R - \bigcup_{\beta \leqq \alpha} P_\beta)$ D has no winning strategy. Because τ^α cannot be a winning strategy of D in ${}_{\boldsymbol{a}^\alpha}\Gamma^S_Q(R - \bigcup_{\beta \leqq \alpha} P_\beta)$ for some play in which D uses τ^α, S wins, and its outcome will be chosen as x^α. It follows from [3, Th. 5.10] that D has no winning strategy in ${}_{a^\alpha}\Gamma_Q^S(R - \cup_{\beta \leqq \alpha} P_\beta)$, but we shall present here a direct proof.

Let A be any set of the first category, and assume that $A = \bigcup_{n<\omega} F_n$, where F_n is nowhere dense. Let $\tau: Q^* \to \{-1, 1\}$ be any strategy for D, and $\boldsymbol{a}$ any con. We shall construct a play $s = \langle s_n: n < \omega \rangle \in Q^\omega$ and a nested sequence $\langle g_n: n < \omega \rangle$ of open intervals such that

(i) s obeys $\boldsymbol{a}$.

(ii) $g_n \cap F_n = \emptyset$

(iii) $s = \lim s_n \in \bigcap g_n$.

Pick an open interval g_0 of measure a_0 such that $g_0 \cap F_0 = \emptyset$ and let s_0 be any rational member of g_0. Assume by induction that s_i, g_i are already defined for $0 < i < n$ so that $s_i \in g_i \subset g_{i-1}$ and that $s_{n-1} \in g = g_{\boldsymbol{a}}(\langle s_0, \cdots, s_{n-1} \rangle)$ (see [3, Definition 5.0]). Let d be the distance from s_{n-1} to $R - g$ and let g_n^1 be a subinterval of $(s_{n-1}, s_{n-1} + d)$ such that $g_n^1 \cap F_n = \emptyset$.

Now consider $g' = \{s_{n-1} - x: s_{n-1} + x \in g_n^1\}$. This is an open subinterval of g. Let g_n^{-1} be a subinterval of g' such that $g^{-1} \cap F_n = \emptyset$. Let $s_n^{-1} \in g_n^{-1} \cap Q$. Finally, put $x_n = s - s_n^{-1}$, $\varepsilon_n = \tau(\langle s_0, \ldots, s_{n-1} \rangle, x_n)$ and $s_n = s_{n-1} + \varepsilon_n x$, $g_n = g^{\varepsilon_n}$.

It is clear that (i)–(iii) hold.

Now let $X = \{x^\alpha: \alpha < \omega_1\}$. For each perfect nowhere-dense set P, $P = P_\alpha$ for some $\alpha < \omega_1$, hence by (1)

$$X \cap P = X \cap P_\alpha = \{x^\beta: \beta < \omega_1\} \cap P \subseteq \{x^\beta: \beta < \alpha\}.$$

So $X \cap P$ is countable. On the other hand, if for some con $\boldsymbol{a}$ ${}_{\boldsymbol{a}}\Gamma_Q^S(X)$ is a win for D, let his winning strategy be τ. So for some $\alpha < \omega_1$ $(\boldsymbol{a}, \tau) = (\boldsymbol{a}^\alpha, \tau^\alpha)$. But then S can play against this strategy so that the outcome is x^α, and the play obeys $\boldsymbol{a}$; as $x^\alpha \in X$, he wins in this play of ${}_{\boldsymbol{a}}\Gamma_Q^S(X)$, a contradiction. So X satisfies both conditions.

3. No countable set can resist D

We shall give now an alternative proof of [1, Th. 2] which states:

THEOREM 3.1. *If X is denumerable then $\Gamma^S(X)$ is a win for D.*

PROOF. Let $\{z_n: n < \omega\}$ be an enumeration of X, so that $z_n \neq z_m$ for $n \neq m$. We shall describe now a strategy for D which tells him how to move, consulting a certain finite function f_n that he changes and extends during the game. f_n will be a function defined for $i < n$, whose values are positive numbers that will satisfy:

(*) $f_{n+1}(i) = f_n(i)$ or else $f_{n+1}(i) < \frac{1}{2} f_n(i)$, $\quad i < n$

(**) for every $m < \omega$, $z_i - f_n(i) \neq z_m$ and $z_i + f_n(i) \neq z_m$.

τ is defined by induction as follows. Put $f_0 = \emptyset$. Suppose that $s_0, \cdots, s_n, x_n$ are already played, and also f_n is already defined. D has now to choose $\varepsilon_n \in \{-1, 1\}$ and thereby he determines $s_{n+1} = s_n + \varepsilon_n x_n$.

a) If an $\varepsilon \in \{-1, 1\}$ exists so that for all $i < n$, $s_n + \varepsilon x_n \notin (z_i - f_n(i), z_i + f_n(i))$, make ε_n equal such an ε.

b) Otherwise, put $i_\varepsilon^n = \min\{i: s_n + \varepsilon x_n \in (z_i - f_n(i), z_i + f_n(i)\}$ and pick ε_n so that $i_{\varepsilon_n}^n \geqq i_{-\varepsilon_n}^n$.

D turns now to define f_{n+1} ensuring:

0) $z_i - f_{n+1}(i)$, $z_i + f_{n+1}(i) \notin X$, $0 \leqq i \leqq n$.

1) If $|s_{n+1} - z_i| \geqq f_n(i)$ then $f_{n+1}(i) = f_n(i)$, $0 \leqq i < n$.

2) If $0 < |s_{n+1} - z_i| \leqq f_n(i)$ then $f_{n+1}(i) \leqq \frac{1}{2}|s_{n+1} - z_i|$, $0 \leqq i \leqq n$.

3) If $s_{n+1} = z_i$, $i < n$, then $f_{n+1}(i) \leqq \frac{1}{2} f_n(i)$. If $s_{n+1} = z_n$, $f_{n+1}(n) = 1$.

It is seen that for $0 \leqq i \leqq n$, $|s_{n+1} - z_i| \geqq f_{n+1}(i)$, unless it happens that $s_{n+1} = z_i$.

Let $s = \langle s_n : n < \omega \rangle$ be a play where D followed this strategy. Let $\langle f_n : n < \omega \rangle$ be its accompanied sequence of finite functions. Assume that s is a convergent sequence, and that $\lim s_n = z_m \in X$. We shall derive a contradiction.

The following is easy to verify, using (*), (1), (2). z_i is an accumulation point of a play s where τ is used $\Leftrightarrow f_n(i)$ changes its value infinitely often $\Leftrightarrow \inf_n f_n(i) = 0$.

Hence, if s is convergent to z_m, then for all $i < m$, $f_n(i)$ is eventually constant. Thus, pick n_1 so that for $n \geqq n_1 \geqq m$, $f_n(i) = f_{n_1}(i) = a_i > 0$ for all $i < m$.

It follows by (1), (2) that for $n \geqq n_1$, $|s_{n+1} - z_i| \geqq a_i > 0$, $0 \leqq i < m$, hence also $|z_m - z_i| \geqq a_i$. By (**), z_m is not an endpoint of any of the intervals $(z_i - a_i, z_i + a_i)$, $0 \leqq i < m$. Hence, there is a positive number d such that $(z_m - d, z_m + d) \cap (z_i - a_i, z_i + a_i) = \emptyset$ for $0 \leqq i < m$. Since $z_m = \lim s_n$, there is an $n_2 \geqq n_1$ such that $|z_m - s_n| < \frac{1}{3} d$ for $n \geqq n_2$. We may assume that $|z_m - s_{n_2}| = \delta > 0$ (otherwise take $n_2 + 1$). Now, for $n \geqq n_2$, $x_n < \frac{2}{3} d$ and $s_n \in (z_m - \frac{1}{3} d, z_m + \frac{1}{3} d)$; hence, $s_n + \varepsilon x_n \notin \bigcup_{i<m} (z_i - a_i, z_i + a_i)$ for any $\varepsilon \in \{-1, 1\}$. It follows that $i_\varepsilon^n \geqq m$ for such an n, ε. So ε_n is so chosen by τ to keep $s_{n+1} = s_n + \varepsilon_n x_n$ away from $(z_n - f_n(m), z_m + f_n(m))$. But this means that $|s_n - z_m| \geqq \delta > 0$ for $n \geqq n_2$.

References

1. A. Ehrenfeucht and G. Moran, *Size direction games over the real line. I*, Israel J. Math. **14** (1973), 163–168.

2. D. A. Martin, and R. M. Solovay, *Internal Cohen extensions*, Ann. Math. Logic **2** (1970), 143–178.

3. G. Moran, *Size direction games over the real line. II*, Israel J. Math. **14** (1973), 418–441.

4. R. M. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. of Math. **92** (1970), 1–56.

THE HEBREW UNIVERSITY OF JERUSALEM
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